POWER DISSIPATION AND KINETIC RELATIONS ON VELOCITY-DISCONTINUITY SURFACES IN COMPRESSIBLE RIGID-PLASTIC MATERIAL

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Wave surfaces of strong velocity discontinuity are considered in an isotropic compressible rigid-plastic material.

For associated laws of plastic flow [1, 2] formulas are derived for power dissipation; continuity is proved of the components of the stress tensor and bounds are established for velocity discontinuities on those surfaces.

The obtained formulas are applied when extrusion is considered of a compressible material from a container through a smooth wedge-like unit.

1. A three-dimensional rigid-plastic body is considered with plasticity condition given by

$$S_{ij}S_{ij} = 2k^2 \quad (S_{ij} = \sigma_{ij} - {}^1/_3\sigma_{kk}\delta_{ij}) \tag{1.1}$$

where σ_{ij} is the stress tensor, S_{ij} is the stress-tensor deviator, k is the physical constant of the material, δ_{ij} is the Kronecker symbol. It is assumed that for the material under consideration the following function exists:

$$e = \varphi(\sigma) \quad (e = {}^1/_3 e_{kk},\ \sigma = {}^1/_3\sigma_{kk}) \tag{1.2}$$

In the above e_{ij} are the components of the deformation tensor.

It follows from the generalized Mises theorem [1] that

$$\varepsilon_{ij}' = \lambda S_{ij} \quad (\lambda = H/2k,\ \varepsilon_{ij}' = \varepsilon_{ij} - {}^1/_3\varepsilon_{kk}\delta_{ij}) \tag{1.3}$$

where ε_{ij} is the tensor of the deformation rate, ε_{ij}' is the deviator of the deformation-rate tensor, $H = (2\varepsilon_{ij}'\varepsilon_{ij}')^{1/2}$ is the intensity of the displacement deformation rates.

The rate ε of volume change appearing in the physical equations (1.3) is found by differentiating the function (1.2) with respect to time, that is,

$$\varepsilon = \frac{d\varphi}{d\sigma}\frac{d\sigma}{dt} \tag{1.4}$$

2. It is assumed by us that the material with properties described above has a surface Σ given by equation $f(x_i, t) = 0$ on which the displacement rates v_i suffer discontinuity.

Let us denote by v_0 the component of the displacement rate of the surface Σ in the direction of its outer normal n and by v_n the projection of the velocity vector of particles of the medium on that normal.

Since stationary discontinuities in velocity are inconsistent with the continuity assumption of the medium, it is assumed that $v_n \neq v_0$. It is supposed that the material moves across the surface Σ with velocity $v_n > v_0$.

It is known [3] that on the surface Σ the deformation rates satisfy the relations

$$\varepsilon_{ij} = \psi([v_i]n_j + [v_j]n_i) \tag{2.1}$$

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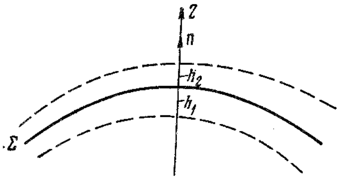

Fig. 1

In the above ψ is a proportionality multiplier, $[v_i] = v_i^+ - v_i^-$ is the difference between the values of velocity taken on each side of the surface Σ, n_i are the projections of the unit vector of the normal to that surface on the rectangular Cartesian coordinate axes x_i.

A local system of coordinates (x, y, z) is introduced at any point of the surface Σ (Fig. 1) in such a way that the normal n is directed along the z axis.

One then obtains for the direction cosines

$$n_x = n_y = 0, \quad n_z = 1 \tag{2.2}$$

It follows from the relations (1.4), (2.1) and (2.2) that

$$\varepsilon_{xx} = \varepsilon_{xy} = \varepsilon_{yy} = 0, \quad \varepsilon_{xz} = \psi [v_x] \neq 0$$
$$\varepsilon_{yx} = \psi [v_y] \neq 0, \quad \varepsilon_{zz} = 2\psi [v_z] \neq 0 \tag{2.3}$$

Hence it follows that in a compressible material side by side with discontinuities in the velocity components which are tangential to the surface Σ, a discontinuity may also occur in its normal component.

Let us assume that on the surface Σ there is a discontinuity of stress σ_{ij}.

The contacting stresses must be continuous on the surface, that is,

$$[\sigma_{ij}]n_j = 0 \quad ([\sigma_{ij}] = \sigma_{ij}^+ - \sigma_{ij}^-) \tag{2.4}$$

In the local coordinate system (2.2) the relations (2.4) become the equalities

$$[\sigma_{xz}] = [\sigma_{yz}] = [\sigma_{zz}] = 0 \tag{2.5}$$

It follows from the relations (2.3) and (2.5) that on the surface Σ one has

$$[\sigma_{ij}]\varepsilon_{ij} = 0 \tag{2.6}$$

For the fluidity condition (1.1) it follows from the maximum principle of dissipation rate of mechanical energy that

$$[\sigma_{ij}]\, \varepsilon_{ij} > 0 \tag{2.7}$$

Comparing (2.6) and (2.7) one can see that on the surface Σ there is the equality

$$[\sigma_{ij}] = 0$$

The latter indicates that for convex fluidity conditions on the surface of velocity discontinuity the stresses are continuous in a compressible material.

The power dissipation for a thin transition layer in which the velocities v_i suffer rapid though continuous changes is given by

$$D = \int_\omega \sigma_{ij}\varepsilon_{ij}\, d\omega = \int_\Sigma \int_{-h_1}^{h_2} \sigma_{ij}\varepsilon_{ij}\, dz\, d\Sigma \tag{2.8}$$

where ω denotes the volume of the infinitely thin layer.

Since the stresses σ_{ij} are continuous on the surface Σ, the hydrostatic pressure is also continuous on Σ, and consequently it is constant across the layer thickness.

Using the latter as well as the relations (1.1) and (1.2) one finds from the formula (2.8) that

$$D = k \int_\Sigma \int_{-h_1}^{h_2} (2\varepsilon_{ij}'\varepsilon_{ij}')^{1/2}\, dz\, d\Sigma + 3 \int_\Sigma \sigma \int_{-h_1}^{h_2} \varepsilon\, dz\, d\Sigma \tag{2.9}$$

It is known [4] that the deformation process in many cases can be described by

$$(v_x - v_x^-)\frac{\partial v_y}{\partial z} = (v_y - v_y^-)\frac{\partial v_x}{\partial z}$$
$$(v_y - v_y^-)\frac{\partial v_z}{\partial z} = (v_z - v_z^-)\frac{\partial v_y}{\partial z} \tag{2.10}$$
$$(v_z - v_z^-)\frac{\partial v_x}{\partial z} = (v_x - v_x^-)\frac{\partial v_z}{\partial z}$$

Solving the above three differential equations (2.10) for the unknown functions v_x, v_y, v_z one finds

$$v_x = C + \varkappa(z),\ v_y = C_1 + C_2\varkappa(z),\ v_z = C_3 + C_1\varkappa(z) \tag{2.11}$$

where the constants C, C_1, C_3 are given as follows:

$$C = v_x^-,\ C_1 = v_y^-,\ C_3 = v_z^-$$

The conditions on the boundaries of the layer are given by the equalities

$$\varkappa(-h_1) = 0,\ \varkappa(h_2) = [v_x],\ C_2\varkappa(h_2) = [v_y],\ C_4\varkappa(h_2) = [v_z] \tag{2.12}$$

By using the relations (2.3), (2.11) and (2.12) a formula is obtained from (2.9) for power dissipation on the surface of velocity discontinuity in a compressible rigid-plastic material:

$$D = \frac{k}{\sqrt{3}}\int_\Sigma \{3(v_x^+ - v_x^-)^2 + 3(v_y^+ - v_y^-)^2 + 4(v_z^+ - v_z^-)^2\}^{1/2} d\Sigma + \int_\Sigma \sigma[v_z]\, d\Sigma \tag{2.13}$$

3. A deformable material is considered whose limit state is given by

$$S_{ij}S_{ij} = 2(k - \theta\sigma)^2 \qquad (\sigma \leqslant k/\theta) \tag{3.1}$$

In the above θ is the physical-mechanical constant of the material.

The associated law of plastic flow is assumed to be of the form [2]

$$\varepsilon_{ij} = \lambda'(\theta\delta_{ij} + S_{ij}/2\sqrt{S_{ij}S_{ij}/2}) \quad (\lambda' = \sqrt{\varepsilon_{ij}\varepsilon_{ij}/(3\theta^2 + 1/2)}) \tag{3.2}$$

Employing the relations (2.8), (3.1), and (3.2) one can represent the power dissipation for a thin transition layer on the surface Σ by the expression

$$D = \frac{k}{(3\theta^2 + 1/2)^{1/2}}\int_\Sigma\int_{-h_1}^{h_2} (\varepsilon_{ij}\varepsilon_{ij})^{1/2} dz\, d\Sigma \tag{3.3}$$

From the last expression, using (2.3), (2.11) and (2.12), one obtains

$$D = \frac{k}{(6\theta^2 + 1)^{1/2}}\int_\Sigma \{(v_x^+ - v_x^-)^2 + (v_y^+ - v_y^-)^2 + 2(v_z^+ - v_z^-)^2\}^{1/2} d\Sigma \tag{3.4}$$

The formulas (2.13) and (3.4) are a generalization of the familar expressions [5, 6] for power dissipation on surfaces of velocity discontinuity to the cases of deformable compressible materials.

It is noted that the formulas (2.13) and (3.4) were obtained for two types of plastic compressibility of the material of quite a different type.

The compressibility which appears in the derivation of the formula (2.13) is fully determined by experiments on even, uniform compression. The compressibility used in the derivation of the formula (3.4) is related basically to the deformation of the material and cannot be obtained from the experiments on even, uniform compression.

In the case of planarly deformable compressible material the plasticity condition (3.1) assumes the form [2]

$$f = 1/2(\sigma_{11} + \sigma_{22})\sin\varphi^\circ + \sqrt{(\sigma_{11} - \sigma_{22})^2/4 + \sigma_{12}^2} - c^\circ\cos\varphi^\circ = 0$$
$$\left(c^\circ = \frac{k}{(1 - 12\theta^2)^{1/2}},\quad \sin\varphi^\circ = \frac{3\theta}{(1 - 3\theta^2)^{1/2}},\quad \cos\varphi^\circ = \left(\frac{1 - 12\theta^2}{1 - 3\theta^2}\right)^{1/2}\right) \tag{3.5}$$

If one assumes that there exists in the material a curve L of velocity discontinuity then using (3.2) and (3.5) the power dissipation for a thin transition layer on the curve L can be given by

$$D = \frac{c^\circ\cos\varphi^\circ}{(1 + \sin^2\varphi^\circ)^{1/2}}\int_S (2\varepsilon_{xx}^2 + 2\varepsilon_{yy}^2 + 4\varepsilon_{xy}^2)^{1/2} dS \tag{3.6}$$

In the above S denotes the area of the infinitely thin layer.

By extending our derivation of the formulas (2.13) and (3.4) to the case of planarly deformable material an expression is obtained from the formula (3.6) for power dissipation on the discontinuity curve L:

$$D = \frac{c^\circ\cos\varphi^\circ}{(1 + \sin^2\varphi^\circ)^{1/2}}\int_L \{(v_x^+ - v_x^-)^2 + 2(v_y^+ - v_y^-)^2\}^{1/2} dL \tag{3.7}$$

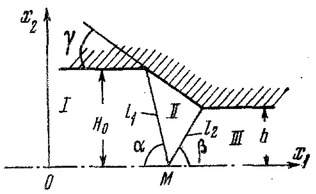

Fig. 2

4. Following [7] the relations will be established on the surface of velocity discontinuity which form constraints on the possible discontinuities of the velocity vector arising in a flow of particles of a compressible material across the surface Σ.

To this end, the relations (2.1) are multiplied by n_j. Then one finds

$$\psi\,[v_i] = \varepsilon_{ik}n_k - \psi\,[v_j]n_jn_i \tag{4.1}$$

Substituting the value of the difference $[v_i]$ from (4.1) into the relations (2.1) and bearing in mind that the material is compressible one finds

$$\varepsilon_{ij} = \varepsilon_{ik}n_jn_k + \varepsilon_{jk}n_in_k - \varepsilon_{kk}n_in_j \tag{4.2}$$

Let $x_i = x_i(t_\alpha)(i = 1, 2, 3)$ be parametric equations of the surface Σ and $t_\alpha(\alpha = 1, 2)$ the curvilinear coordinates on that surface. Then the partial derivatives of the velocity v_i with respect to the coordinates x_j can be written as [8]

$$\frac{\partial v_i}{\partial x_j} = \frac{dv_i}{dn}n_j + g^{\alpha\beta}\frac{\partial v_i}{\partial t_\alpha}\frac{\partial x_j}{\partial t_\beta} \tag{4.3}$$

where dv_i/dn is the derivative of the velocity v_i along the normal n to the surface Σ, and $g^{\alpha\beta}$ $(\alpha, \beta = 1, 2)$ is the contravariant metric tensor of the surface Σ.

Substituting the values of the partial derivatives $\partial v_i/\partial x_j$ in (4.3) into the Cauchy relations

$$\varepsilon_{ij} = {}^1/_2\ (v_{i,j} + v_{j,i})$$

one obtains

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{dv_i}{dn}n_j + \frac{dv_j}{dn}n_i\right) + \frac{1}{2}g^{\alpha\beta}\left(\frac{\partial v_i}{\partial t_\alpha}\frac{\partial x_j}{\partial t_\beta} + \frac{\partial v_j}{\partial t_\alpha}\frac{\partial x_i}{\partial t_\beta}\right) \tag{4.4}$$

Comparing in the latter expressions the subscripts i and j one finds

$$\frac{dv_i}{dn}n_i = \varepsilon_{kk} - g^{\alpha\beta}\frac{\partial v_i}{\partial t_\alpha}\frac{\partial x_i}{\partial t_\beta} \tag{4.5}$$

Using the relations (4.4) and (4.5) one obtains for the derivative dv_i/dn the expression

$$\frac{dv_i}{dn} = 2\varepsilon_{ik}n_k + g^{\alpha\beta}\left(\frac{\partial v_j}{\partial t_\alpha}\frac{\partial x_j}{\partial t_\beta}n_i - \frac{\partial v_k}{\partial t_\alpha}\frac{\partial x_i}{\partial t_\beta}n_k\right) - \varepsilon_{kk}n_i \tag{4.6}$$

The values of dv_i/dn from (4.6) are now inserted into the relations (4.4).

Then using the relations (4.2) and after some transformations one obtains a system of differential equations which must be satisfied by the components of the velocity vector on the surface Σ:

$$g^{\alpha\beta}\left(\frac{\partial v_i}{\partial t_\alpha}\frac{\partial x_j}{\partial t_\beta} + \frac{\partial v_j}{\partial t_\alpha}\frac{\partial x_i}{\partial t_\beta}\right) = g^{\alpha\beta}\frac{\partial v_k}{\partial t_\alpha}n_k\left(\frac{\partial x_i}{\partial t_\beta}n_j + \frac{\partial x_j}{\partial t_\beta}n_i\right) \tag{4.7}$$

The relations (4.7) are multiplied by $\partial x_i/\partial t_\tau$, $\partial x_j/\partial t_\sigma$.

Then using the familiar identity [8]

$$g^{\alpha\beta}\frac{\partial x_i}{\partial t_\alpha}\frac{\partial x_j}{\partial t_\beta} = \delta_{ij} - n_in_j$$

one obtains on the surface Σ three independent relations which at the discontinuities are of the form

$$\frac{\partial\,[v_i]}{\partial t_\sigma}\frac{\partial x_i}{\partial t_\tau} + \frac{\partial\,[v_j]}{\partial t_\tau}\frac{\partial x_j}{\partial t_\sigma} = 0 \tag{4.8}$$

The relations (4.8) represent constraints on the possible changes in the discontinuities of the components of the velocity vector of the displacement on the surface Σ.

5. As an example of the application of the obtained relations (3.7) and (4.8) we consider a setting process of extrusion of a compressible rigid-plastic material from a container through a smooth wedge-like unit (Fig. 2).

Considering the lines l_1 and l_2 as lines of velocity discontinuities it is assumed that in thin transition layers which represent the discontinuity lines l_1 and l_2 the material undergoes during its plastic flow an intensive change of volume and having passed these lines it moves like a rigid body in the regions II and III.

It is also assumed that the material does not suffer any plastic deformation in the region I.

In accordance with our assumptions a kinematically admissible velocity field is selected in the regions I, II and III respectively of the form

$$v_1 = a_1,\ v_2 = 0;\ v_1 = a_2 \cos\gamma,\ v_2 = -a_2 \sin\gamma;\ v_1 = a_3,\ v_2 = 0 \tag{5.1}$$

In the above a_1 is the rate of material supply at the deformation center.

Since the material flows from the region I to the region II and from the region II to the region III we determine the power dissipation on the discontinuity lines l_1 and l_2.

To this end we denote on the lines l_1 and l_2 the normal velocity component by v_n and the tangential by v_t.

On the discontinuity line l_1 the jump differences for the normal and tangential velocity components are

$$\begin{aligned}[v_n] &= \{(a_2\cos\gamma - a_1)\,\mathrm{tg}\,\alpha - a_2\sin\gamma\}\cos\alpha \\ [v_t] &= \{(a_2\cos\gamma - a_1)\,\mathrm{ctg}\,\alpha + a_2\sin\gamma\}\sin\alpha\end{aligned} \tag{5.2}$$

By inserting (5.2) into the formula (3.7) and by integrating one obtains the power dissipation on the l_1 line:

$$\begin{aligned}D_1 = \frac{H_0}{\sin\alpha}\frac{c^\circ\cos\varphi^\circ}{(1+\sin^2\varphi^\circ)^{1/2}}\{[(a_2\cos\gamma - a_1)\,\mathrm{ctg}\,\alpha + a_2\sin\gamma]^2\sin^2\alpha + \\ + 2\,[(a_2\cos\gamma - a_1)\,\mathrm{tg}\,\alpha - a_2\sin\gamma]\cos^2\alpha\}^{1/2}\end{aligned} \tag{5.3}$$

For the tangential and normal velocity components on the discontinuity line l_2 one has

$$\begin{aligned}[v_n] &= \{(a_2\cos\gamma - a_3)\,\mathrm{tg}\,\beta + a_2\sin\gamma\}\cos\beta \\ [v_t] &= \{(a_3 - a_2\cos\gamma)\,\mathrm{ctg}\,\beta + a_2\sin\gamma\}\sin\beta\end{aligned} \tag{5.4}$$

Substituting the corresponding values of (5.4) into (3.7) and integrating one finds

$$\begin{aligned}D_2 = \frac{h}{\sin\beta}\frac{c^\circ\cos\varphi^\circ}{(1+\sin^2\varphi^\circ)^{1/2}}\{[(a_3 - a_2\cos\gamma)\,\mathrm{ctg}\,\beta + a_2\sin\gamma]^2\sin^2\beta + \\ + 2\,[(a_2\cos\gamma - a_3)\,\mathrm{tg}\,\beta + a_2\sin\gamma]^2\cos^2\beta\}^{1/2}\end{aligned} \tag{5.5}$$

By setting the origin of the Cartesian coordinate system $x_1 0 x_2$ at the point M one obtains parametric equations of the discontinuity lines l_1 and l_2, respectively:

$$x_1 = k_1 t,\quad x_2 = k_2 t;\quad x_1 = k_3 t,\quad x_2 = k_4 t \tag{5.6}$$

In the above t is an arbitrary parameter, $k_m (m = 1, 2, 3, 4)$ are constants which satisfy the conditions

$$k_1 / k_2 = -\,\mathrm{tg}\,\alpha,\ k_3 / k_4 = \mathrm{tg}\,\beta \tag{5.7}$$

The kinematic relations (4.8) together with (5.6) and (5.7) take the form

$$[v_1] - [v_2]\,\mathrm{tg}\,\alpha = b_1,\quad [v_1] + [v_2]\,\mathrm{tg}\,\beta = b_2 \tag{5.8}$$

To determine the arbitrary constants b_1 and b_2 one uses the obvious condition which follows from the symmetry of the problem and which imposes constraints on the velocity discontinuities at the point M:

$$[v_1] = a_3 - a_1,\quad [v_2] = 0 \tag{5.9}$$

From (5.9) one obtains

$$b_1 = b_2 = a_3 - a_1 \tag{5.10}$$

Using the relations (5.1), (5.8) and (5.10) one finds

$$\begin{aligned}a_2 &= a_1 / (\cos\gamma - \sin\gamma\,\mathrm{tg}\,\beta) \\ a_3 &= a_1 (\cos\gamma + \sin\gamma\,\mathrm{tg}\,\alpha) / (\cos\gamma - \sin\gamma\,\mathrm{tg}\,\beta)\end{aligned} \tag{5.11}$$

If one disregards friction on the container walls, then in view of the fact that the power of the external and internal forces is the same, one finds

$$Pa_1 = 2(D_1 + D_2) \tag{5.12}$$

where P is the extrusion force.

Using the relations (5.3), (5.5) and (5.11) one determines from (5.12) the magnitude of the extrusion force:

$$P = \frac{2C^\circ \cos\varphi^\circ}{(1+\sin^2\varphi^\circ)^{1/2}(\operatorname{ctg}\gamma - \operatorname{tg}\beta)}\{H_0[(1-\operatorname{tg}\beta\operatorname{ctg}\alpha)^2 + 2(\operatorname{tg}^2\beta + \operatorname{ctg}^2\alpha)]^{1/2} + h[(\operatorname{tg}\alpha\operatorname{ctg}\beta - 1)^2 + 2(\operatorname{ctg}^2\beta + \operatorname{tg}^2\alpha)]^{1/2}\} \tag{5.13}$$

From geometrical considerations (Fig. 2) it is not difficult to obtain that

$$\operatorname{ctg}\alpha = \frac{H_0 - h}{H_0}\operatorname{ctg}\gamma - \frac{h}{H_0}\operatorname{ctg}\beta \tag{5.14}$$

The relation (5.14) can be employed to eliminate the angle α from the formula (5.13) and to use the minimum extrusion force to determine the angle β.

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